A Novel Approach to Non–Hermitian Random Matrix Models

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Abstract

In this paper we propose a new method for studying spectral properties of the non-hermitian random matrix ensembles. Alike *complex* Green's function encodes, via discontinuities, the real spectrum of the hermitian ensembles, the proposed here *quaternion* extension of the Green's function leads directly to complex spectrum in case of non-hermitian ensembles and encodes additionally some spectral properties of the eigenvectors. The standard two-by-two matrix representation of the quaternions leads to generalization of so-called matrix-valued resolvent, proposed recently in the context of diagrammatic methods [1, 2, 3, 4, 5, 6]. We argue that quaternion Green's function obeys Free Variables Calculus [7, 8]. In particular, the quaternion functional inverse of the matrix Green's function, called after [9] Blue's function obeys simple addition law, as observed some time ago [1, 3]. Using this law we derive new, general, algorithmic and efficient method to find the non–holomorphic Green's function for *all* non-hermitian ensembles of the form $H + iH'$, where ensembles H and H' are independent (free in the sense of Voiculescu [7]) hermitian ensembles from arbitrary measure. We demonstrate the power of the method by a straightforward rederivation of spectral properties for several

examples of non–hermitian random matrix models.
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1 Introduction

Random matrix models provide a powerful framework for modeling numerous physical phenomena, with applications covering all branches of theoretical physics [11, 10, 12]. Among different classes of random matrix models the non-hermitian random ensembles form a fascinating class. Contrary to the hermitian ensembles, where real eigenvalues form cuts on real axis, general non-hermitian ensembles develop spectrum, which covers two-dimensional, often multiple-connected support on the whole complex plane. From mathematical point of view, non-hermitian ensembles are challenging, since several standard methods of hermitian random matrix calculus fail in this case. On the other side, the issue of non-hermitian random matrix ensembles is far from being academic. Non-hermitian ensembles are omnipotent in various branches of physics and in interdisciplinary sciences. Sample applications are: open chaotic scattering [13], spectral properties of Euclidean Dirac operators in the presence of chemical potential [14] or CP-violating angle θ in Quantum Chromodynamics [15], non-hermitian generalizations of Anderson localization in mesoscopic systems [16], modeling of the chemical transitions in dissipative systems [17], matrix generalizations of multiplicative diffusion processes [18] or evolution of the spectral curves for non-hermitian ensembles in the context of the growth problem [19]. All these applications call for new calculational methods to deal with the problem of complex spectra.

In this paper we present a new technique, introducing the quaternion generalization of the Green's function, the quaternion generalization of its functional inverse (quaternion Blue's function) and we formulate the quaternion addition law for non-hermitian ensembles. In section 2, we briefly remind basic known facts on hermitian and non-hermitian Green's functions. In section 3, we make a connection to Free Random Variables calculus. We comment also on some previous approaches based on diagrammatic techniques. In section 4, after establishing the notation, we outline the construction and we present the quaternion generalization of the hermitian Green's function and its properties. In particular, we present a general form of the addition law for non-hermitian ensembles.

In section 5, we formulate the operational form of the addition algorithm. In section 6, we adapt the operational form of the addition formalism for

the particular case of the Gaussian randomness. Then, to demonstrate the power of the method, we provide straightforward derivations of three classical results in non-hermitian random matrix models. Section 7 summarizes our conclusions. Appendices hide some necessary technical details and/or proofs.

2 Hermitian and non-hermitian Green's functions

Let us first focus on a hermitian random matrix model. A basic tool to investigate eigenvalues' distribution for H is a resolvent, or Green's function,

$$G_H(z) = \frac{1}{N}\langle \mathrm{Tr}\frac{1}{z1_N - H}\rangle. \tag{1}$$

where 1_N is a unit matrix of size N. The usefulness of this function stems from the fact, that, for any finite N, $G_H(z)$ can be written as

$$G_H(z) = \frac{1}{N}\langle \sum_{i=1}^{N} \frac{1}{z-\lambda_i}\rangle, \qquad \lambda_i \in \mathbb{R}, \tag{2}$$

after diagonalizing H by a unitary similarity transformation, so that $G_H(z)$ is a meromorphic function with N poles at λ_i on the real line. In the large N limit the poles merge into cuts on the real axis.

Green's function $G_H(z)$ can be used to reconstruct the spectral density function $\rho_H(\lambda)$ due to the relation

$$\frac{1}{\lambda + i\epsilon} = \mathrm{PV}\frac{1}{\lambda} - i\pi\delta(\lambda) \tag{3}$$

where PV denotes the principal value distribution and ϵ is meant implicitly approaching limit $\epsilon \to 0$, so that

$$\rho_H(\lambda) = \frac{1}{N}\langle \mathrm{Tr}\delta(\lambda 1_N - H)\rangle = -\frac{1}{\pi}\mathrm{Im}G_H(\lambda + i\epsilon). \tag{4}$$

Hence $\rho_H(\lambda)$ can be read out from discontinuities of the imaginary part of the Green's function.

There are several ways of calculating Green's functions for hermitian random matrix models [10, 11, 12]. Here we mention the diagrammatic approach, after [20]. A starting point of the approach is the expression allowing for the reconstruction of the Green's function from all the moments.

$$G_H(z) = \sum_{n\geq 0} \frac{m_{H,n}}{z^{n+1}}, \qquad m_{H,n} = \frac{1}{N}\langle \mathrm{Tr}H^n\rangle. \tag{5}$$

The resolvent $G_H(z)$ is interpreted as a series in the guise of the Feynman-like diagrammatic expansion in the large number of colors limit. This series can be efficiently evaluated exploiting the analogy of RMT to $0+0$ dimensional gauge field theory in the 't Hooft large N number of colors limit [23]. To avoid unnecessary repetitions, we refer to the literature [2, 20].

The reason why the above procedure works correctly for *hermitian* matrix models is the fact that the Green's function is guaranteed to be *holomorphic* in the whole complex plane except at most on one or more 1-dimensional intervals.

The key difference which arises in the non–hermitian case (we denote the general non-hermitian matrix by X) is that eigenvalues of X are complex in general; in the large N limit they form two–dimensional domains in the complex plane, in contrary to one–dimensional cuts in the previous case. Therefore the power series expansion (5) no longer captures the full information about the Green's function. In particular the eigenvalue distribution is related to the non-analytic (non-holomorphic) behavior of the Green's function:

$$\frac{1}{\pi}\partial_{\bar{z}}G(z) = \rho(z)\ . \tag{6}$$

This phenomenon can be easily seen even in the the simplest non-hermitian ensemble — the Ginibre-Girko one [21, 22], with non-hermitian matrices X, and measure

$$P(X) = e^{-N\mathrm{Tr}XX^\dagger}\ . \tag{7}$$

It is easy to verify that all moments vanish $\langle \mathrm{tr}\, X^n\rangle = 0$, for $n > 0$ so the expansion (5) gives the Green's function to be $G(z) = 1/z$. The true answer is, however, different. Only for $|z| > 1$ one has indeed $G(z) = 1/z$. For $|z| < 1$ the Green's function is nonholomorphic and equals $G(z) = \bar{z}$.

The above difficulty was first addressed in mathematical papers. Brown [24] defined a measure for complex ensembles as

$$\mu_X = \frac{1}{2\pi}\left(\frac{\partial^2}{(\partial\Re\lambda)^2} + \frac{\partial^2}{(\partial\Im\lambda)^2}\right)\log\det(X-\lambda 1_N) \tag{8}$$

where

$$\det(X-\lambda 1_N) = \exp\left[\frac{1}{N}\mathrm{Tr}\log\sqrt{(X-\lambda 1_N)(X^\dagger-\bar{\lambda}1_N)}\right] \tag{9}$$

is known in mathematics as Fuglede-Kadison determinant. For some recent results on Brown measure we refer to [25, 26].

Physicists have addressed the problem of measure, exploiting the analogy to two-dimensional electrostatics [13, 22, 27]. Let us define the "electrostatic potential"

$$\begin{aligned} F &= \frac{1}{N}\mathrm{Tr}\ln[(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N]\,. \\ &= \frac{1}{N}\ln\det[(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N] \end{aligned} \tag{10}$$

Then

$$\begin{aligned} \lim_{\epsilon\to 0}\frac{\partial^2 F(z,\bar{z})}{\partial z\partial\bar{z}} &= \lim_{\epsilon\to 0}\frac{1}{N}\left\langle \mathrm{Tr}\frac{\epsilon^2}{(|z1_N - X|^2 + \epsilon^2 1_N)^2}\right\rangle \\ &= \frac{\pi}{N}\left\langle \sum_i \delta^{(2)}(z-\lambda_i)\right\rangle \equiv \pi\rho(x,y) \end{aligned} \tag{11}$$

represents Gauss law, where $z = x + iy$. The last equality follows from the representation of the complex Dirac delta

$$\pi\delta^{(2)}(z-\lambda_i) = \lim_{\epsilon\to 0}\frac{\epsilon^2}{(\epsilon^2 + |z-\lambda_i|^2)^2} \tag{12}$$

In the spirit of the electrostatic analogy we can define the Green's function $G(z,\bar{z})$, as an "electric field"

$$G \equiv \frac{\partial F}{\partial z} = \frac{1}{N}\lim_{\epsilon\to 0}\left\langle \mathrm{Tr}\frac{\bar{z}1_N - X^\dagger}{(\bar{z}1_N - X^\dagger)(z1_N - X) + \epsilon^2 1_N)}\right\rangle. \tag{13}$$

Then Gauss law leads (6).

The drawback of the above construction is that due to the quadratic structures in the denominator, form (13) is difficult to use in practical calculation.

Instead of working *ab initio* with the object (13), one can take consider the following generalization [2]. One defines the matrix-valued resolvent through

$$\begin{aligned} \hat{\mathcal{G}} &= \frac{1}{N}\left\langle \mathrm{bTr}_2\begin{pmatrix} z1_N - X & i\epsilon 1_N \\ i\epsilon 1_N & \bar{z}1_N - X^\dagger\end{pmatrix}^{-1}\right\rangle = \\ &= \frac{1}{N}\left\langle \mathrm{bTr}_2\begin{pmatrix} A & B \\ C & D\end{pmatrix}\right\rangle \equiv \begin{pmatrix} \mathcal{G}_{11} & \mathcal{G}_{1\bar{1}} \\ \mathcal{G}_{\bar{1}1} & \mathcal{G}_{\bar{1}\bar{1}}\end{pmatrix} \end{aligned} \tag{14}$$

with

$$A = \frac{\bar{z}1_N - X^\dagger}{(\bar{z}1_N - X^\dagger)(z1_N - X) + \epsilon^2 1_N}$$

$$
\begin{aligned}
B &= \frac{-i\epsilon}{(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N} \\
C &= \frac{-i\epsilon}{(\bar{z}1_N - X^\dagger)(z1_N - X) + \epsilon^2 1_N} \\
D &= \frac{z1_N - X}{(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N}
\end{aligned} \tag{15}
$$

and where we introduced the 'block trace' defined as

$$
\mathrm{bTr}_2 \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{2N\times 2N} \equiv \begin{pmatrix} \mathrm{Tr}\ A & \mathrm{Tr}\ B \\ \mathrm{Tr}\ C & \mathrm{Tr}\ D \end{pmatrix}_{2\times 2}. \tag{16}
$$

Then, by definition, the upper-right component $\mathcal{G}_{11}$, hereafter denoted by $G_X(z,\bar{z})$, is equal to the Green's function (13).

The block approach has several advantages. First of all it is *linear* in the random matrices X allowing for a simple diagrammatic calculational procedure. Let us define $2N$ by $2N$ matrices

$$
Z_\epsilon = \begin{pmatrix} z1_N & i\epsilon 1_N \\ i\epsilon 1_N & \bar{z}1_N \end{pmatrix} \quad , \quad \mathcal{H} = \begin{pmatrix} X & 0 \\ 0 & X^\dagger \end{pmatrix}. \tag{17}
$$

Then the generalized Green's function is given formally by the same definition as the usual Green's function G,

$$
\mathcal{G} = \frac{1}{N} \left\langle \mathrm{bTr}_2 \frac{1}{Z_\epsilon - \mathcal{H}} \right\rangle . \tag{18}
$$

What is more important, also in this case the Green's function is completely determined by the knowledge of all matrix-valued moments

$$
\left\langle \mathrm{bTr}_2 \ \ {Z_\epsilon}^{-1} \mathcal{H} {Z_\epsilon}^{-1} \mathcal{H} \ldots {Z_\epsilon}^{-1} \right\rangle . \tag{19}
$$

This last observation allowed for a diagrammatic interpretation [2]. The Feynman rules were analogous to the hermitian ones, only now one has to keep track of the block structure of the matrices.

Let us summarize the general properties [2] of the matrix valued generalized Green's function. Each component of the matrix carries important information about the stochastic properties of the system. There are always two solutions for $\mathcal{G}_{11}(X)$, one holomorphic, another non-holomorphic. The second one leads, via Gauss law, to the eigenvalue distribution. The first one, the holomorphic one is not "spurious", it represents the generating function for the real *moments* of the complex distribution [28], but, by definition, cannot reproduce the non-holomorphic spectrum of the ensemble.

The shape of the "coastline" bordering the "sea" of complex eigenvalues is determined by the matching conditions for the two solutions, i.e. it is determined by imposing on the non-holomorphic solution the condition $\mathcal{G}_{12}\mathcal{G}_{21} = 0$. The off-diagonal elements have an interesting interpretation [29]. They represent the correlator between the left and right *eigenvectors* (introduced in [30])

$$\left\langle \sum_a (L_a|L_a)(R_a|R_a)\delta(z-\lambda_a) \right\rangle = -\frac{N}{\pi}\mathcal{G}_{12}\mathcal{G}_{21} \equiv -\frac{N}{\pi}C_X(z,\bar{z}) \qquad (20)$$

On boundary of the domain of the eigenvalues, the above correlator vanishes.

We would like to mention for completeness, that another, similar approach appears often in the literature, under the "hermitization method" name [3, 4, 5]. Basically it uses an alternative representation for the determinant in (10), which is rewritten in a hermitian form

$$\det[(z1_N - X)(\bar{z}1_N - X^\dagger) + \epsilon^2 1_N] = -\det \begin{pmatrix} 1_N & z1_N - X \\ \bar{z}1_N - X^\dagger & -\epsilon^2 1_N \end{pmatrix} \qquad (21)$$

Both versions lead to the similar results.

An important feature of the explained above method of the generalized matrix-valued Green's functions is their link to Free Random Variables calculus [7, 8], which we will address in the next section.

3 Free Random Variables Approach

In the hermitian RMT a method of free random variables (FRV), introduced by Voiculescu [7], is widely used [9, 31, 32, 33]. The power of the method lies in the fact, that in particular it allows to find the resolvent for the sum of two independent (free) [1] ensembles H_1 and H_2, on the basis of knowledge of the resolvents for each separated ensemble only. In other words, operation R introduced by Voiculescu linearizes the non-commutative, matrix convolution of hermitian random variables, alike the logarithm of characteristic function does it for sum of two identical, independent, one-dimensional random variables.

In this paper we recall a version of Free Random Variables formulated by Zee [9] and known under the name *Blue's functions*. Blue's function is simply related to original R-function of Voiculescu

$$B(z) = R(z) + \frac{1}{z} \qquad (22)$$

[1] Freeness is a non-commutative analog of statistical independence, introduced by Voiculescu [7].

and its usefulness comes from the fact, that it is simply the functional inverse of the resolvent

$$B(G(z)) = G(B(z)) = z. \tag{23}$$

The algorithm of "adding" ensembles is therefore straightforward: consider two freely independent hermitian random matrices, H and H', and their (holomorphic) Green's functions, $G_H(z)$ and $G_{H'}(z)$. The addition algorithm goes as follows: First, one inverts functionally $G_H(z)$ and $G_{H'}(z)$ to get Blue's functions $B_H(z)$ and $B_{H'}(z)$. Second, using the addition law

$$B_{H+H'}(z) = B_H(z) + B_{H'}(z) - \frac{1}{z}. \tag{24}$$

one obtains the Blue's function for the sum of ensembles. Third, inverting functionally this function one recovers needed $G_{H+H'}(z)$.

The generalization of the concept of the Blue's function for non-hermitian ensembles was first proposed in [1] and confirmed by [3, 4]. The generalized, matrix-valued Blue's function was defined as a functional inverse of matrix-valued Green's function, i.e.

$$\mathcal{G}(\mathcal{B}(Z_\epsilon)) = Z_\epsilon \tag{25}$$

for matrix Z (17). Such defined Blue's functions fulfilled the similar, but now matrix-valued addition law, i.e.

$$\mathcal{B}_{1+2}(Z) = \mathcal{B}_1(Z) + \mathcal{B}_2(Z) - Z^{-1} \tag{26}$$

with ϵ safely put to zero. The explicit form of the matrix-valued Blue's function for an arbitrary ensembles was, however, far from obvious. In several applications using this method, an additional insight coming from e.g. diagrammatic interpretation was necessary.

One of the purposes of this paper is to propose the rigorous method yielding the matrix-valued Blue's function solely on the basis of algebraic properties, and abandoning any additional insight from e.g. diagrammatization methods. We will shown in the following chapters, that such construction is not only possible, but leads to strikingly simple algorithms for addition of a broad class of non-hermitian ensembles.

4 Quaternion Green's functions

The main advantage of working with the complex Green's function for hermitian ensembles stems from the holomorphic properties of the resolvent. This function, being holomorphic everywhere except the intervals on real

axis, allows, by analytical calculation, to recover the spectral function by approaching the cuts from above and from below, i.e. by calculating the discontinuities along the cuts.

It is tempting to propose the similar method in the case of complex spectra. A natural generalization is the algebra of quaternions. Such a speculation appeared in the literature [4], but the explicit realization of such construction was never completed and left as a challenge. In this chapter we propose the solution. We will now proceed in the following way:

1. First, we introduce the notation.

2. Second, we generalize the notion of a matrix–valued Green's function (18) by exchanging Z_ϵ by a general quaternion Q thus exploiting the whole four–dimensional quaternion space and not only its part 'close' to the complex plane. A quaternion Blue's function is its functional inverse.

3. Third, we compute quaternion Green's and Blue's functions for a hermitian random matrix H.

4. Fourth, we investigate how these functions behave during multiplying a general random matrix by a fixed complex number g.

5. Fifth, we present an addition formula which enables to express a quaternion Blue's function of a sum of two freely independent general random matrices by their respective quaternion Blue's functions. Then we compute a quaternion Blue's function for a non-hermitian $X = H + iH'$, H and H' being two hermitian and freely independent random matrices.

4.1 Notation

We found it convenient to introduce the following notation:
Writing down explicitly any square $N \times N$ matrix we add a subscript '$_N$' to make the reader sure about the matrix' size. The blank places in a matrix mean that they are occupied by zeroes. We use standard Pauli matrices. If X is a general complex $N \times N$ matrix then

$$X^{\mathrm{D}} = \begin{pmatrix} X & \\ & X^\dagger \end{pmatrix}_{2N}, \tag{27}$$

where the superscript '$^{\mathrm{D}}$' means '<u>d</u>uplication' since we duplicate X into X and $X^\dagger$. For quaternions, i. e. combinations of 1_2 and three Pauli matrices of the form

$$Q = x_0 1_2 + i\vec{x}\cdot\vec{\sigma}, \qquad x_0, x_1, x_2, x_3 \in \mathbb{R} \tag{28}$$

we use a 2×2 matrix notation

$$Q = \begin{pmatrix} a & i\bar{b} \\ ib & \bar{a} \end{pmatrix}_2, \qquad a, b \in \mathbb{C} \tag{29}$$

with $a = x_0 + ix_3, b = x_1 + ix_2$. If Q is a general quaternion, then

$$Q^{\mathrm{U}} = \begin{pmatrix} a1_N & i\bar{b}1_N \\ ib1_N & \bar{a}1_N \end{pmatrix}_{2N}, \tag{30}$$

where the superscript 'U' means 'uniatal extension', since we extend Q by multiplying each of its elements by a unit matrix (N is a fixed number defined below) and also

$$Q^{\mathrm{I}} = Q \begin{pmatrix} i & \\ & -i \end{pmatrix}_2 = Qi\sigma_3 = \begin{pmatrix} ia & \bar{b} \\ -b & -i\bar{a} \end{pmatrix}_2, \tag{31}$$

where the superscript 'I' comes from imaginary units appearing in (31). A quaternion of a special interest for us is defined when a complex variable z is being considered,

$$Z = \begin{pmatrix} z & \\ & \bar{z} \end{pmatrix}_2. \tag{32}$$

Generally, quaternion Green's (Blue's) functions we denote by calligraph alphabet to avoid confusion with complex standard Green's (Blue's) functions $G(z)$ $(B(z))$.

4.2 Definition of a Quaternion Green's Function

Let us consider a random $N \times N$ matrix X, hermitian or not. We see that the key quantity for X is its matrix–valued Green's function (18); it has the form of a complex Green's function with a complex variable z substituted by a $2N \times 2N$ matrix Z_ϵ^{U} and X substituted by a $2N \times 2N$ matrix X^{D}.

Let us consider a straightforward generalization of (18): from now on we call *the quaternion Green's function* the object

$$\mathcal{G}_X(Q) = \frac{1}{N} \langle \mathrm{bTr}_2 \frac{1}{Q^{\mathrm{U}} - X^{\mathrm{D}}} \rangle, \tag{33}$$

where Q is a quaternion

$$Q = \begin{pmatrix} a & i\bar{b} \\ ib & \bar{a} \end{pmatrix}_2. \tag{34}$$

This defines $\mathcal{G}_X$ as a quaternion function of a quaternion variable.

The only difference to the original meaning of this notion is that we exchanged Z_ϵ with a general quaternion Q thus promoting $\mathcal{G}_X$ to be *the function of* Q. Particularly, for $Q = Z_\epsilon$ we arrive at the original meaning [1]. The key idea of this work is that $\mathcal{G}_X$ has a few new interesting features *as a function of* Q which cannot be seen when regarding only the previous case of $Q = Z_\epsilon$, i.e. assuming that ϵ is infinitesimally small.

The first interesting feature of $\mathcal{G}_X$ is that it enables to define *the matrix Blue's function* as the *functional* inverse of $\mathcal{G}_X$:

$$\mathcal{G}_X(\mathcal{B}_X(Q)) = \mathcal{B}_X(\mathcal{G}_X(Q)) = Q, \tag{35}$$

for any Q. This is clearly the quaternion function of the quaternion variable. It is a natural generalization from the complex case (23). Note that if Q is a quaternion, then $\mathcal{G}_X(Q)$ and $\mathcal{B}_X(Q)$ are quaternions too. It requires $\mathcal{G}_X$ as a function of Q everywhere and not only 'near' Z. It is particularly visible when studying the *eigenvalues* of Q. A quaternion Q (28) has a very simple structure of eigenvalues,

$$q_1 \equiv q = x_0 + i|\vec{x}|, \qquad q_2 = \bar{q} = x_0 - i|\vec{x}|. \tag{36}$$

The eigenvalues have following basic properties:

- They are mutually conjugated.
- They are different (there is no degeneracy) always except of a very special case of $\vec{x} = 0$, which means that the only degenerate quaternion is the one of the form $Q = x_0 1_2$. In other words, they are non–real always except the degenerate case.
- $|q|^2 = x_0^2 + \vec{x}^2 = |a|^2 + |b|^2 = \mathrm{Det}Q$. Since the determinant is positive, it always fulfills the requirement of Fuglede-Kadison construction for Brown measure.

Let us finish with speculative remarks. Introducing quaternions seems to be very natural from the algebraic point of view, however the reader will find by inspection, that for applications presented here not the whole quaternion space is necessary; it can be restricted to the subspace with $x_2 = 0$. Since our calculations do not simplify considerably with regards to $x_2 = 0$ case, we assume the general case $x_2 \neq 0$. Finally, we mention that the construction presented here holds if instead of quaternions a general 2×2 matrix (biquaternion) is used. In this case the matrix-space is eight-dimensional, however some nice properties of quaternion eigenvalues are lost.

4.3 Quaternion Green's and Blue's functions for a Hermitian Random Matrix

Consider a known hermitian random $N \times N$ matrix H. We show how to calculate quaternion Green's and Blue's functions for hermitian H. The method is based upon a simple behavior of quaternion Green's functions under similarity transformations (39) and a knowledge of their form on diagonal matrices (37). The same holds for Blue's functions, (40) and (38).

Let us first note, that a quaternion Green's function for H at the point of a *diagonal matrix* has a very simple form of a diagonal matrix with holomorphic Green's functions for H on the diagonal

$$\mathcal{G}_H(\begin{pmatrix} q & \\ & \bar{q} \end{pmatrix}_2) = \begin{pmatrix} G_H(q) & \\ & G_H(\bar{q}) \end{pmatrix}_2. \tag{37}$$

Applying $\mathcal{B}_H$ to both sides of (37) and exchanging q, $\bar{q}$ with $G_H(q)$, $G_H(\bar{q})$ we get an identical theorem for a quaternion Blue's function for H:

$$\mathcal{B}_H(\begin{pmatrix} q & \\ & \bar{q} \end{pmatrix}_2) = \begin{pmatrix} B_H(q) & \\ & B_H(\bar{q}) \end{pmatrix}_2. \tag{38}$$

To regain a general quaternion Q from its eigenvalues we use a similarity transformation S. The quaternion Green's function for H cooperates very well with similarity transformations of its argument; consider a similarity transformation S that is an internal operation in the quaternion space, then

$$\mathcal{G}_H(S^{-1}QS) = S^{-1}\mathcal{G}_H(Q)S. \tag{39}$$

The similar formula holds for a quaternion Blue's function for H,

$$\mathcal{B}_H(S^{-1}QS) = S^{-1}\mathcal{B}_H(Q)S. \tag{40}$$

Proofs are presented in appendix A.

From these two properties we can also deduce that eigenvalues of $\mathcal{G}_H(Q)$ (resp. $\mathcal{B}_H(Q)$) are $(G_H(q), G_H(\bar{q}))$ (resp. $(B_H(q), B_H(\bar{q}))$).

These two simple algebraic properties of quaternion Green's and Blue's functions, (37), (39) and (38), (40) allow us to calculate them for any Q. Consider now any non–degenerate quaternion $Q = \begin{pmatrix} a & i\bar{b} \\ ib & \bar{a} \end{pmatrix}_2$ with eigenvalues q, $\bar{q}$ and let S be the similarity transformation diagonalizing Q,

$$Q = S^{-1}\begin{pmatrix} q & \\ & \bar{q} \end{pmatrix}_2 S. \tag{41}$$

It is straightforward to apply expressions (37) and (39) to find a general form of $\mathcal{G}_X(Q)$. For a given Q we compute an explicit form of S and S^{-1}, then a simple three–matrix multiplication leads to the result. The result is derived in appendix B:

$$\mathcal{G}_H(Q) = \gamma_H(q,\bar{q})1_2 - \gamma'_H(q,\bar{q})Q^\dagger, \tag{42}$$

where γ_H and γ'_H are two scalar functions depending only on Q's eigenvalues and are given by

$$\gamma_H(q,\bar{q}) = \frac{qG_H(q) - \bar{q}G_H(\bar{q})}{q-\bar{q}}, \tag{43}$$

$$\gamma'_H(q,\bar{q}) = \frac{G_H(q) - G_H(\bar{q})}{q-\bar{q}}. \tag{44}$$

The degenerate case of $Q = x_0 1_2$ is trivial since the degenerate quaternion is already diagonal, so that from (37)

$$\mathcal{G}_H(x_0 1_2) = G_H(x_0)1_2. \tag{45}$$

This is the solution; we have expressed $\mathcal{G}_H$ by G_H, i.e. the quaternion generalization using the known, holomorphic resolvent.

Let us note, that due to (38) and (40) the identical construction can be made for $\mathcal{B}_H$; hence the analogous expression for non–degenerate quaternion Q,

$$\mathcal{B}_H(Q) = \beta_H(q,\bar{q})1_2 - \beta'_H(q,\bar{q})Q^\dagger, \tag{46}$$

where β_H and β'_H are similar to γ_H and γ'_H but with B_H in the place of G_H,

$$\beta_H(q,\bar{q}) = \frac{qB_H(q) - \bar{q}B_H(\bar{q})}{q-\bar{q}}, \tag{47}$$

$$\beta'_H(q,\bar{q}) = \frac{B_H(q) - B_H(\bar{q})}{q-\bar{q}}. \tag{48}$$

If a degeneracy is present, a formula analogous to (45) holds. Again, we have expressed quaternion $\mathcal{B}_H$ by standard analytic B_H.

4.4 Quaternion Green's and Blue's functions for a General Random Matrix Multiplied by a Fixed Complex Number

We have already computed quaternion Green's and Blue's functions for a hermitian random matrix H. Now there is time for the passage to a non–hermitian random matrix. The following problem is the introductory one

before doing so: consider a general random matrix X and a fixed complex number g. We want to express the quaternion Green's and Blue's functions for gX through the ones for X.

If $g = 0$ then trivially $\mathcal{G}_0(Q) = \frac{1}{Q}$, so let us further assume that $g \neq 0$. In appendix C we present a simple derivation of

$$\mathcal{G}_{gX}(Q) = \mathcal{G}_X(\begin{pmatrix} 1/g & \\ & 1/\bar{g} \end{pmatrix}_2 Q) \begin{pmatrix} 1/g & \\ & 1/\bar{g} \end{pmatrix}_2. \tag{49}$$

This would be incorrect unless the argument were just a quaternion Q; in fact $\begin{pmatrix} 1/g & \\ & 1/\bar{g} \end{pmatrix}_2 Q$ is a quaternion with

$$a \longrightarrow a/g, \qquad b \longrightarrow b/\bar{g} \tag{50}$$

(see (29)). In terms of variables x_i, (see (28)), this replacement is non–trivial.

For a special case of real g, Eq. (49) simplifies,

$$\mathcal{G}_{gX}(Q) = \frac{1}{g}\mathcal{G}_X(\frac{1}{g}Q), \qquad g \in \mathbb{R} \setminus \{0\}. \tag{51}$$

Let us note that this formula is identical to the one well known for a complex Green's function and complex g,

$$G_{gX}(z) = \frac{1}{g}G_X(\frac{1}{g}z). \tag{52}$$

Similar behavior appears at the level of Blue's functions. Again, if $g = 0$, then trivially $\mathcal{B}_0(Q) = \frac{1}{Q}$, so let us further assume that $g \neq 0$. From the definition (35) and (49) we immediately get

$$\mathcal{B}_{gX}(Q) = \begin{pmatrix} g & \\ & \bar{g} \end{pmatrix}_2 \mathcal{B}_X(Q \begin{pmatrix} g & \\ & \bar{g} \end{pmatrix}_2). \tag{53}$$

The argument here is a quaternion with

$$a \longrightarrow ag, \qquad b \longrightarrow b\bar{g}. \tag{54}$$

For a special case of real g Eq. (53) simplifies,

$$\mathcal{B}_{gX}(Q) = g\mathcal{B}_X(gQ), \qquad g \in \mathbb{R} \setminus \{0\}. \tag{55}$$

Let us note that this formula is identical to the one for a complex Blue's function and complex g,

$$B_{gX}(z) = gB_X(gz). \tag{56}$$

4.5 A Quaternion Blue's Function for a Non–Hermitian Random Matrix with Hermitian and Anti–Hermitian Parts Freely Independent

In this subsection we apply a general formalism of the previous part to a particular problem: consider a non hermitian random $N \times N$ matrix X of the form

$$X = H + iH', \tag{57}$$

where the (hermitian) random matrices H and H' are freely independent. The advantage of quaternion Blue's functions stems from the fact, that, not-surprisingly, they obey the addition law (the proof parallels the proof of an "addition law" for hermitized matrices in [3], so we are not repeating it here)

$$\mathcal{B}_X(Q) = \mathcal{B}_H(Q) + \mathcal{B}_{iH'}(Q) - \frac{1}{Q}. \tag{58}$$

We can now easily compute the quaternion Blue's function for X assuming the knowledge of quaternion Blue's functions for H and H', since they are expressed using (46), i.e. their complex Blue functions.

Using (46) and (53), we may rewrite (58) in an explicit form,

$$\mathcal{B}_X(Q) = \beta_H(q,\bar{q})1_2 + \beta_{H'}(q^{\mathrm{I}},\overline{q^{\mathrm{I}}}) \begin{pmatrix} i & \\ & -i \end{pmatrix}_2 -$$

$$- [\beta'_H(q,\bar{q}) + \beta'_{H'}(q^{\mathrm{I}},\overline{q^{\mathrm{I}}}) + \frac{1}{\mathrm{Det}Q}]Q^{\dagger}, \tag{59}$$

where q^{I}, $\overline{q^{\mathrm{I}}}$ are eigenvalues of Q^{I}, see (31).

This explicit form of the addition law for any ensemble of the form $H+iH^{'}$ (where H and $H^{'}$ are free) is one of the main results of this paper.

Few comments are helpful.

- The $^{\mathrm{I}}$'s terms came from the multiplication formula (53) and the non–real number i before $H^{'}$. Note that the presence of i in (57) is essential for the non–hermiticity and it is i that brings Q^{I}, so the existence of the eigenvalues of Q^{I} on top of eigenvalues of Q is exactly a sign of non–hermiticity.

- The operation $^{\mathrm{I}}$ changes the trace of Q and does not change its determinant,

$$\mathrm{Tr}Q^{\mathrm{I}} = i(a-\bar{a}) = -2x_3 \neq 2x_0 = \mathrm{Tr}Q, \qquad \mathrm{Det}Q^{\mathrm{I}} = \mathrm{Det}Q, \tag{60}$$

see (31), so that $\mathrm{Tr}Q^{\mathrm{I}}$ cannot be expressed through Q's invariants. Since in general

$$q = x_0 + i|\vec{x}| = x_0 + i\sqrt{x_1^2 + x_2^2 + x_3^2}, \tag{61}$$

then

$$q^{\mathrm{I}} = -x_3 + i\sqrt{x_1^2 + x_2^2 + x_0^2}, \tag{62}$$

and it is impossible to express q^{I} only by q. Note that the eigenvalues are related since

$$|q| = |q^{\mathrm{I}}|. \tag{63}$$

- The algebraic properties (37), (38) of quaternion Green's and Blue's functions for hermitian random matrices remain valid for general random matrices,

$$\mathcal{B}_X(\begin{pmatrix} q & \\ & \bar{q} \end{pmatrix}_2) = \begin{pmatrix} B_X(q) & \\ & B_{X^\dagger}(\bar{q}) \end{pmatrix}_2 = \begin{pmatrix} B_X(q) & \\ & \overline{B_X(q)} \end{pmatrix}_2, \tag{64}$$

and identically for $\mathcal{G}_X$. (We use general formulae, $B_{X^\dagger}(\bar{q}) = \overline{B_X(q)}$ and identically for G.) It is important to note, that due to the presence of q^{I},

$$\mathcal{B}_X(S^{-1}QS) \neq S^{-1}\mathcal{B}_X(Q)S \tag{65}$$

and similar relation holds for $\mathcal{G}_X$. Indeed, this important non–trivial behavior under the similarity transformation is precisely a footprint of non–hermiticity of X. The immediate conclusion is that ($B_X(q)$ and $\overline{B_X(q)}$) (resp. ($G_X(q)$ and $\overline{G_X(q)}$)) *are not* the eigenvalues of $\mathcal{B}_X(Q)$ (resp. $\mathcal{G}_X(Q)$).

5 Holomorphic and Non–Holomorphic Green's Functions for a Non–Hermitian Random Matrix with Hermitian and Anti–Hermitian Parts Freely Independent

In this section, we invert functionally in a fixed point of $Q = Z$ which gives $\mathcal{G}_X(Z)$ and the (complex) non–holomorphic Green's function for X in particular. The result is given by a surprisingly simple and operationally convenient formula. We have already computed the quaternion Blue's function for $X = H + iH'$. We should now invert it functionally to get the quaternion

Green function for X, however this is in fact unnecessary since in this paper we are interested only in $\mathcal{G}_X(Z)$. Hence we need only to solve the equation

$$\mathcal{B}_X(\begin{pmatrix} A & i\bar{B} \\ iB & \bar{A} \end{pmatrix}_2) = \begin{pmatrix} z & \\ & \bar{z} \end{pmatrix}_2, \tag{66}$$

where we denote $\mathcal{G}_X(Z) \equiv \begin{pmatrix} A & i\bar{B} \\ iB & \bar{A} \end{pmatrix}_2$. We use it to extract $A = G_X(z, \bar{z})$ and $-|B|^2 = C_X(z, \bar{z})$.

One may ask why it is allowed to put here the regulator $\epsilon = 0$. Note first that $\mathcal{G}_X(Q)$ is discontinuous in $Q = Z$, i. e. $\mathcal{G}_X(Z) \neq \mathcal{G}_X(Z_\epsilon)$. On the other hand, we will show that there are always two solutions of (66), the first one gives $\mathcal{G}_X(Z)$ and the second one, $\mathcal{G}_X(Z_\epsilon)$. Finally, the answer is that *functional inverting of a function in her discontinuity point gives two solutions, the first one is a value of the function in this point and the second one is a limit value of the function when approaching the discontinuity point.* In other word, the limit $\epsilon \to 0$ is precisely encoded in the functional inverting.

Rewrite (66) using (59),

$$\begin{pmatrix} k + ik' - l\bar{A} & li\bar{B} \\ liB & k - ik' - lA \end{pmatrix}_2) = \begin{pmatrix} z & \\ & \bar{z} \end{pmatrix}_2, \tag{67}$$

where for short

$$k = \beta_H(\mathcal{G}_X(Z)), \tag{68}$$

$$k' = \beta_{H'}(\mathcal{G}_X(Z)^{\mathrm{I}}), \tag{69}$$

$$l = \beta'_H(\mathcal{G}_X(Z)) + \beta'_{H'}(\mathcal{G}_X(Z)^{\mathrm{I}}) + \frac{1}{\mathrm{Det}\mathcal{G}_X(Z)}. \tag{70}$$

Looking at the two off–diagonal equations we see that this equation has always two solutions, one with $B = 0$ and one with $B \neq 0$.

Let us infer the first case. For $B = 0$, $\mathcal{G}_X(Z)$ is diagonal, hence from (64)

$$Z = \mathcal{B}_X(\begin{pmatrix} A & \\ & \bar{A} \end{pmatrix}_2) = \begin{pmatrix} B_X(A) & \\ & \overline{B_X(A)} \end{pmatrix}_2, \tag{71}$$

which leads to

$$A = G_X(z). \tag{72}$$

We may call this solution the *holomorphic* one since it is expressed by a holomorphic Green's function for X.

The second case is a general one and we call it the *non–holomorphic* one since then A is a non–holomorphic Green's function for X.

To sum it up, *there always exist two solutions of (66), the first one with $B = 0$, which reads $A = G_X(z)$ and which is called the holomorphic solution and the second one with $B \neq 0$ which is called the non–holomorphic solution.*

5.1 The Non–Holomorphic Solution

Since in (68), (69), (70) functions β and β' are expressed via eigenvalues of $\mathcal{G}_X(Z)$ and $\mathcal{G}_X(Z)^{\mathrm{I}}$, let us denote these eigenvalues by g and g^{I}, and their conjugate partners respectively. From subsection (4.5) we know that they are *not* equal to $G_X(z)$, $G_X(\bar{z})$.

In the non–holomorphic case the off–diagonal equations of (66) are

$$l = 0, \tag{73}$$

because it is possible to divide both sides by $i\bar{B}$ or iB; explicitly

$$\frac{B_H(g) - B_H(\bar{g})}{g - \bar{g}} + \frac{B_{H'}(g^{\mathrm{I}}) - B_{H'}(\overline{g^{\mathrm{I}}})}{g^{\mathrm{I}} - \overline{g^{\mathrm{I}}}} + \frac{1}{|g|^2} = 0. \tag{74}$$

Two diagonal equations of (66)

$$\begin{aligned} k + ik' - lD &= z, \\ k - ik' - lA &= \bar{z} \end{aligned}$$

gives due to (73)

$$k = x, \tag{75}$$
$$k' = y, \tag{76}$$

where $z = x + iy$. Explicitly

$$\frac{gB_H(g) - \bar{g}B_H(\bar{g})}{g - \bar{g}} = x, \tag{77}$$
$$\frac{g^{\mathrm{I}}B_{H'}(g^{\mathrm{I}}) - \overline{g^{\mathrm{I}}}B_{H'}(\overline{g^{\mathrm{I}}})}{g^{\mathrm{I}} - \overline{g^{\mathrm{I}}}} = y. \tag{78}$$

The form of these equations suggests introducing two new (complex in general) variables, m and m', thus changing two equations (77), (78) into four equations

$$\begin{aligned} gB_H(g) &= xg + m, \\ \bar{g}B_H(\bar{g}) &= x\bar{g} + m, \\ g^{\mathrm{I}}B_{H'}(g^{\mathrm{I}}) &= yg^{\mathrm{I}} + m', \\ \overline{g^{\mathrm{I}}}B_{H'}(\overline{g^{\mathrm{I}}}) &= y\overline{g^{\mathrm{I}}} + m' \end{aligned}$$

Conjugating the first and third ones, comparing with the remaining two and exploiting $\overline{B_H(z)} = B_H(\bar{z})$ true for hermitian matrices, we get

$$m, m' \in \mathbb{R} \tag{79}$$

and two equations

$$B_H(g) = x + \frac{m}{g}, \tag{80}$$

$$B_{H'}(g^{\mathrm{I}}) = y + \frac{m'}{g^{\mathrm{I}}},. \tag{81}$$

Substituting (80), (81) into the off–diagonal equation (74) gives a simple result,

$$m + m' = 1. \tag{82}$$

To bring the solution to the end we have to express A and B through g and g^{I}; using general expressions (61) and (62) we get

$$A = \Re g - i\Re g^{\mathrm{I}} \tag{83}$$

and

$$-|B|^2 = \frac{1}{2}[((\Re g)^2 + (\Re g^{\mathrm{I}})^2) - ((\Im g)^2 + (\Im g^{\mathrm{I}})^2)]. \tag{84}$$

Finally, we can write down the main result of this section: the algorithm of calculating a non–holomorphic Green function for a non–hermitian random matrix X with hermitian and anti–hermitian parts freely independent.

Let us start with given non–hermitian random matrix $X = H + iH'$, with H and H' freely independent hermitian ensembles. The holomorphic Green's or Blue's functions for H and H' are explicitly known or given implicitly by some equations. The algorithm goes as follows:

1. Write down two equations

$$B_H(g) = x + \frac{m}{g}, \tag{85}$$

$$B_{H'}(g^{\mathrm{I}}) = y + \frac{1-m}{g^{\mathrm{I}}}, \tag{86}$$

   with three unknown quantities, g, g^{I} and $m \in \mathbb{R}$. Find from them $g+\bar{g}$, $g\bar{g}$, $g^{\mathrm{I}} + \overline{g^{\mathrm{I}}}$ and $g^{\mathrm{I}}\overline{g^{\mathrm{I}}}$ expressed via m.

2. Compute m from the third equation,

$$g\bar{g} = g^{\mathrm{I}}\overline{g^{\mathrm{I}}}. \tag{87}$$

3. Put m into $g + \bar{g}$, $g^{\mathrm{I}} + \overline{g^{\mathrm{I}}}$ and $|g|^2$ expressed through m (step 1).

These three steps yield the non–holomorphic Green's function and correlator between left and right eigenvectors for X:

$$G_X(x,y) = \frac{1}{2}[(g+\bar{g}) - i(g^{\rm I} + \overline{g^{\rm I}})], \tag{88}$$

$$C_X(x,y) = \frac{1}{4}(g^2 + \bar{g}^2 + (g^{\rm I})^2 + \overline{g^{\rm I}}^2) = \tag{89}$$

$$= \frac{1}{4}[(g+\bar{g})^2 + (g^{\rm I} + \overline{g^{\rm I}})^2] - |g|^2. \tag{90}$$

The general algorithm contains also a simple method of deriving an equation (in coordinates (x,y)) of a borderline of X's eigenvalues' domains. The borderline is exactly the place where holomorphic and non–holomorphic Green functions meet together. And it is the limit $B \to 0$ that carries the non–holomorphic solution towards its boundary with the holomorphic one. Hence *the borderline's equation* is simply

$$B = 0, \tag{91}$$

where B is an appropriate element of the *non–holomorphic solution*. This is equivalent to the condition of vanishing of the correlator between left and right eigenvectors,

$$C_X(x,y) = 0, \tag{92}$$

where we compute $C_X(x,y)$ for the *non–holomorphic solution*.

Due to (90) and (87), the general equation for the curve defining the support of the eigenvalue domains is given by

$$(g+\bar{g})^2 + (g^{\rm I} + \overline{g^{\rm I}})^2 = 4g\bar{g}, \tag{93}$$

where $g+\bar{g}$, $g^{\rm I} + \overline{g^{\rm I}}$ and $g\bar{g}$ are expressed via m in step 1 and explicitly in step 3, so that this equation binds x and y to form the borderline.

Before closing this section, few comments are useful. First, inferring the equations (85), (86) we look for a solution that comes with its conjugate partner in pair. (E. g., if g satisfies the equation, $\bar{g}$ does it as well.) So that two possibilities are allowed, either this is the pair of two *different* (and hence *non–real*) numbers, or they merge into a *one real* number. The second possibility means actually, that $\mathcal{G}_X(Z)$ is proportional (with real coefficient) to a diagonal matrix, i. e. (a) in case of g, eq. (85), to 1_2 (b) in case of $g^{\rm I}$, eq. (86), to $i\sigma_3$. However, this is impossible since we are looking for a non–holomorphic solution with non–diagonal $\mathcal{G}_X(Z)$. To sum up, *we are looking for a solution that is non–real and comes in pair with its conjugate partner*.

Second, it is not actually our task to find all g's in this step but only their sums, products etc. , e.g. we may use the Viete rules.

Third, note that it is possible to use the holomorphic Green's functions for H or H' rather than Blue's ones in (85), (86), by inverting these equations functionally. In some cases this approach may be considerably simpler.

6 The Version of the Algorithm in the Case of H from GUE

A common case is when one of the ensembles belongs to Gaussian class, but for a second a holomorphic Blue's function is given by a complicated equation. In this case one may reformulate slightly the algorithm.

Here let us deal with the problem of Gaussian H with

$$B_H(s) = s + \frac{1}{s} \tag{94}$$

and *very complicated* $B_{H'}(s)$. If so, equations (85), (86) are very complicated and another way is preferred.

The quaternion Blue's function for H is obtained easily from (46) and reads

$$\mathcal{B}_H(Q) = Q + \frac{1}{Q},$$

hence the equation (66), after short manipulation, becomes

$$\mathcal{B}_{H'}(\begin{pmatrix} iA & \bar{B} \\ -B & -i\bar{A} \end{pmatrix}_2) = \begin{pmatrix} i(A-z) & -\bar{B} \\ B & i(\bar{z}-\bar{A}) \end{pmatrix}_2.$$

The key observation is to invert the last equation functionally.

Let h, $\bar{h}$ be the eigenvalues of $\begin{pmatrix} i(A-z) & -\bar{B} \\ B & i(\bar{z}-\bar{A}) \end{pmatrix}_2$, i. e. from the general formula (61)

$$h = y - \Im A + i\sqrt{(\Re A - x)^2 + |B|^2},$$

where we recall $z = x + iy$.

Hence from (42)

$$\begin{pmatrix} iA & \bar{B} \\ -B & -i\bar{A} \end{pmatrix}_2 = \gamma_{H'}(h,\bar{h})1_2 - \gamma'_{H'}(h,\bar{h}) \begin{pmatrix} i(\bar{z}-\bar{A}) & \bar{B} \\ -B & i(A-z) \end{pmatrix}_2 \tag{95}$$

which is the equation to solve.

The off–diagonal equations of (95) are thus simply

$$\gamma'_{H'}(h,\bar{h}) = \frac{G_{H'}(h) - G_{H'}(\bar{h})}{h-\bar{h}} = -1. \tag{96}$$

The diagonal ones

$$iA = \gamma_{H'}(h,\bar{h}) - \gamma'_{H'}(h,\bar{h})i(\bar{z}-\bar{A}),$$

become due to (96)

$$iA = \gamma_{H'}(h,\bar{h}) + i(\bar{z}-\bar{A}),$$

or equivalently

$$\gamma_{H'}(h,\bar{h}) = \frac{hG_{H'}(h) - \bar{h}G_{H'}(\bar{h})}{h-\bar{h}} = -y, \tag{97}$$

$$A + \bar{A} = x. \tag{98}$$

Note differences between (97), (98) and (77), (78). We solve the second diagonal equation, (98), by introducing a new unknown quantity n,

$$A = \frac{x}{2} + n. \tag{99}$$

The off–diagonal equation (96) is solved by introducing a new unknown variable m,

$$G_{H'}(h) = -h + m, \qquad m \in \mathbb{R}. \tag{100}$$

Inserting this into the first diagonal equation (97) we get easily m,

$$m = h + \bar{h} - y; \tag{101}$$

but $h + \bar{h} = 2y - 2n$, so that

$$m = y - 2n \qquad \longrightarrow \qquad n = \frac{y-m}{2}$$

and hence

$$A = \frac{z - im}{2}. \tag{102}$$

Finally, it is now possible to write down the algorithm of calculating a non–holomorphic Green's function for a non–hermitian random matrix X with hermitian and anti–hermitian parts freely independent and with hermitian part being a GUE random matrix.

We consider a non–hermitian random matrix $X = H + iH'$, with H and H' freely independent and $B_H(s) = s + \frac{1}{s}$, is given. The complex holomorphic Green's or Blue functions for H' is known (or more often) given by some (complicated) equation. The algorithm reads:

1. Write down an equation

$$G_{H'}(h) = -h + m, \tag{103}$$

with two unknown quantities, h and $m \in \mathbb{R}$. Find from it $h + \bar{h}$ and $h\bar{h}$ expressed via m.

2. Compute m from the second equation,

$$h + \bar{h} = y + m. \tag{104}$$

3. (Necessary only for C_X and the borderline's equation.) Put m into the expression for $h\bar{h}$.

The results of these three steps are given explicitly by

$$G_X(x, y) = \frac{z - im}{2}, \tag{105}$$

$$C_X(x, y) = \frac{1}{4}[x^2 + (y + m)^2] - h\bar{h}. \tag{106}$$

The borderline's equation is

$$4h\bar{h} = x^2 + (y + m)^2, \tag{107}$$

where m is known from step 2 and $h\bar{h}$ from step 3.

6.1 Examples

To demonstrate the usefulness of the advocated approach, we recalculate three classical examples well known in the literature. We are not describing these models in detail, neither their physical context. We start with known holomorphic Green's (or Blue's) functions for hermitian and anti–hermitian parts, referring for further details to the original papers.

• **The Girko–Ginibre Model** [21, 22].
Gaussian complex ensemble, known also as a Girko-Ginibre ensemble, corresponds to the case when $X = H + iH'$, where both H and H' are Gaussian Unitary Ensembles. Since the holomorphic Green's function in this case equals simply to $G(z) = (z - \sqrt{z^2 - 4})/2$, with imaginary part yielding a seminal Wigner semicircle [34], the functional inverses of the resolvents are simply

$$B_H(s) = rs + \frac{1}{s}, \qquad B_{H'}(s) = r's + \frac{1}{s}, \tag{108}$$

where r, r' are real and positive constants (arbitrary variances of the Gaussian distributions). We apply the general algorithm:
Step 1: Equations (85), (86) are quadratic

$$rg^2 - xg + 1 - m = 0, \qquad r'(g^{\rm I})^2 - y(g^{\rm I}) + m = 0,$$

hence from Viete rules

$$g + \bar{g} = \frac{x}{r}, \qquad |g|^2 = \frac{1-m}{r},$$
$$g^{\rm I} + \overline{g^{\rm I}} = \frac{y}{r'}, \qquad |g^{\rm I}|^2 = \frac{m}{r'}.$$

Step 2: The equation for m is

$$\frac{1-m}{r} = \frac{m}{r'} \qquad \longrightarrow \qquad m = \frac{r'}{r+r'}.$$

Step 3: We get finally

$$G_X(x,y) = \frac{1}{2}(\frac{x}{r} - i\frac{y}{r'}), \tag{109}$$
$$C_X(x,y) = \frac{1}{4}(\frac{x^2}{r^2} + \frac{y^2}{r'^2}) - \frac{1}{r+r'} \tag{110}$$

or in the special case of $r = r' = 1$,

$$G_X(x,y) = \frac{1}{2}\bar{z}, \tag{111}$$
$$C_X(x,y) = \frac{1}{4}(x^2+y^2) - \frac{1}{2}. \tag{112}$$

Borderline is given by (110)

$$\frac{x^2}{r^2} + \frac{y^2}{r'^2} = \frac{4}{r+r'}, \tag{113}$$

which is the ellipse with semi–axes $\frac{2r}{\sqrt{r^2+r'^2}}$ and $\frac{2r'}{\sqrt{r^2+r'^2}}$. In the case of $r = r' = 1$ this is a circle with radius $\sqrt{2}$. Applying Gauss law to non-holomphic solution yields spectral distribution (here a constant).

•Model of the Chaotic Resonance Scattering [13]
Non-hermitian ensemble $X = H + iH'$ is defined by H belonging to GUE (or GOE) and H' is a Wishart ensemble. Corresponding Blue's functions were e.g. calculated in [1, 29].

$$B_H(s) = s + \frac{1}{s}, \qquad B_{H'}(s) = -\frac{cr}{1+cs} + \frac{1}{s}, \tag{114}$$

where r, c are real constants. We apply the general algorithm.
Step 1:

$$g^2 - xg + 1 - m = 0, \qquad yc(g^{\rm I})^2 + (cr - cm + y)g^{\rm I} - m = 0,$$

hence from Viete rules

$$g + \bar{g} = x, \qquad |g|^2 = 1 - m,$$

$$g^{\rm I} + \bar{g^{\rm I}} = \frac{cm - cr - y}{yc}, \qquad |g^{\rm I}|^2 = -\frac{m}{yc}.$$

Step 2: The equation for m is

$$1 - m = -\frac{m}{yc} \qquad \longrightarrow \qquad m = \frac{yc}{yc - 1}.$$

Step 3 gives

$$g + \bar{g} = x, \qquad g^{\rm I} + \bar{g^{\rm I}} = \frac{c}{yc - 1} - \frac{r}{y} - \frac{1}{c}, \qquad |g|^2 = \frac{1}{1 - yc},$$

thence

$$G_X(x, y) = \frac{1}{2}[x - i(\frac{c}{yc - 1} - \frac{r}{y} - \frac{1}{c})], \tag{115}$$

$$C_X(x, y) = \frac{1}{4}[x^2 + (\frac{c}{yc - 1} - \frac{r}{y} - \frac{1}{c})^2] - \frac{1}{1 - yc}. \tag{116}$$

The borderline's equation is

$$x^2 + (\frac{c}{yc - 1} - \frac{r}{y} - \frac{1}{c})^2 = \frac{4}{1 - yc}. \tag{117}$$

•The Complex Pastur Model
By complex Pastur we mean a class $X = H + iH^{'}$, where H is deterministic and $H^{'}$ belongs to GUE. We choose for simplicity the deterministic part with only two distinct opposite eigenvalues, so

$$G_H(s) = \frac{1}{2}\left[\frac{1}{s - \mu} + \frac{1}{s + \mu}\right], \tag{118}$$

where μ is a real constant. The general algorithm is being applied.
Step 1:

$$g = \frac{x + \frac{m}{g}}{(x + \frac{m}{g})^2 - \mu^2} \qquad \longrightarrow \qquad (x^2 - \mu^2)g^2 + x(2m - 1)x + m(m - 1) = 0,$$

$$(g^{\rm I})^2 - yg^{\rm I} + m = 0,$$

hence from Viete rules

$$g + \bar{g} = \frac{x(1-2m)}{x^2-\mu^2}, \qquad |g|^2 = \frac{m(m-1)}{x^2-\mu^2},$$

$$g^{\rm I} + \bar{g^{\rm I}} = y, \qquad |g^{\rm I}|^2 = m.$$

Step 2: The equation for m reads

$$\frac{m(m-1)}{x^2-\mu^2} = m \qquad \longrightarrow \qquad m = x^2 - \mu^2 + 1 \qquad \text{or} \qquad m = 0.$$

Step 3 gives

$$g + \bar{g} = -2x - \frac{x}{x^2-\mu^2}, \qquad g^{\rm I} + \bar{g^{\rm I}} = y, \qquad |g|^2 = x^2 - \mu^2 + 1$$

or

$$g + \bar{g} = \frac{x}{x^2-\mu^2}, \qquad g^{\rm I} + \bar{g^{\rm I}} = y, \qquad |g|^2 = 0.$$

In the second case the first and third equation contradict each other, so that finally

$$G_X(x,y) = -\frac{iy}{2} - x - \frac{x}{2(x^2-\mu^2)}, \tag{119}$$

$$C_X(x,y) = \frac{1}{4}[(2x + \frac{x}{2(x^2-\mu^2)})^2 + y^2] - (x^2 - \mu^2 + 1). \tag{120}$$

The borderline's equation reads

$$y^2 = 4(x^2 - \mu^2 + 1) - (2x + \frac{x}{2(x^2-\mu^2)})^2 \tag{121}$$

or after some manipulations reducing it to the form calculated first by Stephanov [14][2]

$$y^2 = \frac{-4\mu^2x^4 + x^2(8\mu^4 - 4\mu^2 - 1) + 4\mu^4(1-\mu^2)}{(x^2-\mu^2)^2}. \tag{122}$$

[2]Original calculation involved *chiral* GUE, but in the leading N one-point Green's function does not feel the difference in the bulk of the spectra. Note, however, that two-point Green's functions are different for chiral and non-chiral complex Pastur ensembles [35].

7 Summary

In this paper we introduced the concept of *quaternion* Green's function for non-hermitian ensembles with *complex* spectra, borrowing from the analogy of *complex* Green's function for hermitian ensembles with *real* spectra. The resulting matrix resolvent encodes important spectral properties. First, there are always two solutions for the diagonal elements (e.g. $\mathcal{G}_{11}$). The nonholomorphic one yields, via Gauss law, the spectral density. The second solution is holomorphic. Both solutions match along the spectral curve, defining the support of the eigenvalues. The product of off-diagonal elements gives [29] the correlator between the left and right eigenvectors, hence the shape of the spectral curve can be also inferred by equating the value of this correlator to zero. The functional inverses of the quaternion Green's function, named quaternion Blue's function, obey for statistically independent ensembles an addition law, following precisely similar laws in Free Random Variables calculus for the case of hermitian ensembles. All these results confirm and generalize previous observations made recently in the literature. In particular, construction [1, 2] can be viewed as a special case of the quaternion. The similar construction proposed in [3, 4, 5] can be viewed as a variant of the presented here, with quaternion replaced by a two-by-two *hermitian* matrix. The advantage of the approach presented here lies in the fact, that the defined procedure is purely algebraic, and does not involve any additional implicit insights from e.g. diagrammatic approaches, used in the other works. It is also simpler at the operational level then the hermitization procedure. Last but not least, we dare to say that the quaternion extension in the case of complex spectra is an esthetic construction.

We proposed the general, simple, algebraic algorithm of finding the matrix–valued Green's function for an arbitrary non–hermitian random matrix models with hermitian and anti–hermitian parts freely independent. This reduced the problem to solving some given explicitly (usually polynomial) equations. The resulting algorithm is surprisingly simple, as we hope the reader does infer by comparing three classical examples recalculated here using our method to original derivations.

We concentrated here on mathematical aspects and the clarity of the presentation, postponing announcing new results for particular physical models to the approaching papers.

We considered here only random non-hermitian ensembles with complex entries, not discussing the real or quaternion ensembles. In the case of one-point function and in the leading large N expansion, the Green's functions for similar real or quaternion cases are expected to be similar modulo trivial rescalings. However, challenging problem is the subleading behavior, or

two-point Green's functions (wide-correlators) for nonhermitian ensembles. We expect that an extension of the quaternion technique may considerably simplify the calculations of wide-correlators for two- and more-point Green's functions for non-hermitian ensembles. These ideas will be addressed next.

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A Behavior of Quaternion Green's and Blue's Functions for a Hermitian Random Matrix under Similarity Transformations

In this appendix we present proofs of theorems (39) and (40).

A.1 Proof of (39)

Since H is hermitian,

$$H^{\mathrm{D}} = \begin{pmatrix} H & \\ & H \end{pmatrix}_{2N}. \tag{123}$$

The diagonal structure and the equality of the diagonal elements implies that H^{D} commutes with any matrix of the form S^{U},

$$[H^{\mathrm{D}}, S^{\mathrm{U}}] = 0; \tag{124}$$

it is easy to check. Hence

$$S^{-1,\mathrm{U}} H^{\mathrm{D}} S^{\mathrm{U}} = H^{\mathrm{D}}. \tag{125}$$

Now let us choose S to be a similarity transformation that is internal in the quaternion space, i. e. if Q is a quaternion, then $S^{-1}QS$ too. There exist such transformations as is shown in appendix B.

Now from the definition (33) and this simple property of H^{D}, (125), we have

$$\begin{aligned} \mathcal{G}_H(S^{-1}QS) &= \frac{1}{N}\langle \mathrm{bTr}_2[(S^{-1}QS)^{\mathrm{U}} - H^{\mathrm{D}}]^{-1}\rangle = \\ &= \frac{1}{N}\langle \mathrm{bTr}_2[S^{-1,\mathrm{U}}(Q^{\mathrm{U}} - H^{\mathrm{D}})S^{\mathrm{U}}]^{-1}\rangle = \dots \end{aligned}$$

The S^{U}s can be get out of the block–trace (and the average of course) because of the simple relation for a block–trace,

$$\mathrm{bTr}_2(YM^{\mathrm{U}}) = (\mathrm{bTr}_2 Y)M \tag{126}$$

true for any $2N \times 2N$ matrix Y and 2×2 matrix M. Hence

$$\ldots = S^{-1}\frac{1}{N}\langle \mathrm{bTr}_2(Q^{\mathrm{U}} - H^{\mathrm{D}})^{-1}\rangle S = S^{-1}\mathcal{G}_H(Q)S,$$

what was to prove.

A.2 Proof of (40)

From (39) and from the definition (35) we have

$$\mathcal{G}_H(S^{-1}QS) = S^{-1}\mathcal{G}_H(Q)S$$

$$S^{-1}QS = \mathcal{B}_H(S^{-1}\mathcal{G}_H(Q)S)$$

$$S^{-1}\mathcal{B}_H(Q)S = \mathcal{B}_H(S^{-1}\mathcal{G}_H(\mathcal{B}_H(Q))S) = \mathcal{B}_H(S^{-1}QS),$$

what was to prove.

B Quaternion Green's and Blue's Functions for a Hermitian Random Matrix

In this appendix we present proofs of theorems (42) and (46) and also some additional information.

B.1 Proof of (42) and (46)

The proof comes in three parts, first for a given Q we compute the similarity transformation S which diagonalizes Q, second we invert it to get S^{-1} and third we use these explicit forms to apply (37) and (39) to get the final result.

Let us assume, that $Q = \begin{pmatrix} a & i\bar{b} \\ ib & \bar{a} \end{pmatrix}_2$ is non–degenerate (eigenvalues are $q \neq \bar{q}$) since we know the degenerate case be trivial, $Q = x_0 1_2$. Let us assume also that Q is not diagonal, which means $b \neq 0$, since the opposite case does not need any diagonalizing transformation.

First let us find an explicit expression for S assuming the knowledge of a and b. A short calculation shows that

$$S = \begin{pmatrix} s_1 & s_1\frac{a-q}{-ib} \\ s_2\frac{\bar{a}-\bar{q}}{-ib} & s_2 \end{pmatrix}_2$$

for any complex s_1, s_2. This is a general similarity transformation that is internal in the quaternion space. Let us choose for simplicity e. g. $s_1 = ib$, $s_2 = i\bar{b}$, which gives

$$S = \begin{pmatrix} ib & q-a \\ \bar{q}-\bar{a} & i\bar{b} \end{pmatrix}_2 . \tag{127}$$

(The 21 element $\bar{q}-\bar{a}$ can be exchanged by $a-q$ due to $a+\bar{a} = \mathrm{Tr}Q = q+\bar{q}$.)

A determinant reads after a simple calculation

$$\mathrm{Det}S = (q-a)(q-\bar{q}); \tag{128}$$

it is always non–zero (S is invertible) in the considered situation.

The expression for S^{-1} will obviously be needed; easily

$$S^{-1} = \frac{1}{q-\bar{q}} \begin{pmatrix} \frac{i\bar{b}}{q-a} & -1 \\ 1 & \frac{ib}{q-a} \end{pmatrix}_2 . \tag{129}$$

Finally

$$\mathcal{G}_H(Q) = \mathcal{G}_H(S^{-1} \begin{pmatrix} q & \\ & \bar{q} \end{pmatrix}_2 S) =$$

$$= \frac{1}{q-\bar{q}} \begin{pmatrix} \frac{i\bar{b}}{q-a} & -1 \\ 1 & \frac{ib}{q-a} \end{pmatrix}_2 \begin{pmatrix} G_H(q) & \\ & G_H(\bar{q}) \end{pmatrix}_2 \cdot$$

$$\cdot \begin{pmatrix} ib & q-a \\ \bar{q}-\bar{a} & i\bar{b} \end{pmatrix}_2 = \dots$$

Noticing also $-|b|^2 = (a-q)(\bar{a}-q)$ and multiplying these three matrices we immediately recover the final result (42),

$$\dots = \frac{qG_H(q) - \bar{q}G_H(\bar{q})}{q-\bar{q}} 1_2 - \frac{G_H(q) - G_H(\bar{q})}{q-\bar{q}} Q^\dagger .$$

The identical construction leads to (46).

B.2 Some Expressions Helpful to Cross–Check (42) and (46)

Let us note the following easy to check relations,

$$\gamma_H(B_H(q), B_H(\bar{q})) = \frac{\beta_H(q,\bar{q})}{\beta'_H(q,\bar{q})}, \tag{130}$$

$$\gamma'_H(B_H(q), B_H(\bar{q})) = \frac{1}{\beta'_H(q,\bar{q})} \tag{131}$$

and similarly in the opposite way.

Moreover, all γs and βs are real.

Using these expressions we can e. g. immediately show the cross–check relation $\mathcal{G}_H(\mathcal{B}_H(Q)) = \mathcal{B}_H(\mathcal{G}_H(Q)) = Q$ which ensures us that the derivation is correct.

C Quaternion Green's and Blue's Functions for a General Random Matrix Multiplied by a Fixed Complex Number

From the definition (33) and for complex $g \neq 0$:

$$\mathcal{G}_{gX}(Q) = \frac{1}{N}\langle \mathrm{bTr}_2(Q^{\mathrm{U}} - (gX)^{\mathrm{D}})^{-1}\rangle =$$

$$= \frac{1}{N}\langle \mathrm{bTr}_2(Q^{\mathrm{U}} - \begin{pmatrix} gX & \\ & \bar{g}X^\dagger \end{pmatrix}_{2N})^{-1}\rangle =$$

$$= \frac{1}{N}\langle \mathrm{bTr}_2(Q^{\mathrm{U}} - \begin{pmatrix} g1_N & \\ & \bar{g}1_N \end{pmatrix}_{2N} X^{\mathrm{D}})^{-1}\rangle =$$

$$= \frac{1}{N}\langle \mathrm{bTr}_2((\begin{pmatrix} \frac{1}{g}1_N & \\ & \frac{1}{\bar{g}}1_N \end{pmatrix}_{2N} Q^{\mathrm{U}} - X^{\mathrm{D}})^{-1} \begin{pmatrix} \frac{1}{g}1_N & \\ & \frac{1}{\bar{g}}1_N \end{pmatrix}_{2N})\rangle =$$

$$= \mathcal{G}_X(\begin{pmatrix} 1/g & \\ & 1/\bar{g} \end{pmatrix}_2 Q) \begin{pmatrix} 1/g & \\ & 1/\bar{g} \end{pmatrix}_2 = \ldots$$

this is the formula we were looking for,

$$\ldots = \mathcal{G}_X(\begin{pmatrix} 1/g & \\ & 1/\bar{g} \end{pmatrix}_2 Q) \begin{pmatrix} 1/g & \\ & 1/\bar{g} \end{pmatrix}_2 . \tag{132}$$

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